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ON THE STABILITY OF A TANGENTIAL DISCONTINUITY IN A MAGNETIZABLE MEDIUM

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We consider the stability of a tangential discontinuity in an inviscid, incompressible and non-heat-conducting, magnetizable fluid, with the dependence of the magnetic permeability on temperature and magnetic field taken into account. Solutions of the problem of stability of a tangential discontinuity in the conventional hydrodynamics and magnetohydrodynamics were given in [1 – 4]. The stability of the interface separating two stationary ferromagnetic fluids was studied in [5]. The problem of stability of a tangential discontinuity in a ferromagnetic fluid was solved in [6]. In [5, 6] it was assumed that the magnetic permeability was independent of temperature and magnetic field.

A system of ferrohydrodynamic equations can be written in the form [7, 8]

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla\left(p + \frac{1}{4\pi}\int_0^H \mu H\, dH\right) + \frac{\mu}{8\pi}\nabla H^2, \quad \operatorname{div}\mathbf{v} = 0 \tag{1}$$

$$\frac{d}{dt}\left[s + \frac{1}{4\pi\rho}\int_0^H \left(\frac{\partial\mu}{\partial T}\right)_H H\, dH\right] = 0$$

$$\operatorname{rot}\mathbf{H} = 0, \quad \operatorname{div}\mu\mathbf{H} = 0, \quad \mu = \mu(T, H)$$

where μ is magnetic permeability, $\mathbf{H}$ is magnetic field, T is temperature, s is entropy of unit mass of the medium and the remaining notation is standard. In [9] the inner moment of impulse was taken into account and the permeability μ was assumed independent of the magnetic field, therefore the equations obtained there differ from (1). The coordinate system is chosen so that the plane $z = 0$ represents the unperturbed surface of tangential discontinuity which separates two fluid layers moving in the direction parallel to the plane of discontinuity. The magnetic field acts in an arbitrary direction. All quantities in the region $z < 0$ are denoted by a subscript 1, and those in the region $z > 0$ by the subscript 2.

According to (1), the magnetic field potential is $\mathbf{H} = \nabla\varphi$. The last equation in (1) can be written in the form

$$\operatorname{div}(\mu\nabla\varphi) = 0 \tag{2}$$

In the coordinate system in which the discontinuity is at rest, the boundary conditions at the surface of tangential discontinuity are

$$\{\varphi\} = 0, \quad \{\mu\mathbf{n}\nabla\varphi\} = 0, \quad \{\Pi_{ik}\, n_i n_k\} = \alpha / R, \quad v_n = 0 \tag{3}$$
$$(\{a\} = a_1 - a_2)$$

Here $\mathbf{n}$ is the unit vector normal to the surface of discontinuity, $1 / R$ is the curvature of the surface, α denotes the surface tension and Π_{ik} is the impulse flux density tensor of the form

$$\Pi_{ik} = \rho v_i v_k + p\delta_{ik} - \frac{\mu H_i H_k}{4\pi} + \frac{\delta_{ik}}{4\pi}\int_0^H \mu H\, dH$$

The energy flux and the tangential stresses are identically equal on both sides of the discontinuity. This follows from the first three boundary conditions (3) and from the condition of continuity of the tangential component of the electric field.

We shall investigate the stability of the tangential discontinuity in the linear approximation. Linearizing the system of equations (1) and (2) and the boundary conditions (3), we obtain

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}_0\nabla\right)\mathbf{v}' = -\nabla(p' + mT'), \qquad \operatorname{div}\mathbf{v}' = 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0\nabla\right)\left[qT' + \frac{1}{4\pi}\mu_T \mathbf{H}_0\nabla\varphi'\right] = 0$$

$$\mu_0\Delta\varphi' + a(\mathbf{H}_0\nabla)^2\varphi' + \mu_T\mathbf{H}_0\nabla T' = 0, \quad \{H_{0z}\zeta + \varphi'\} = 0$$

$$\left\{-\mu_0\mathbf{H}_0\nabla\zeta + \mu_0\frac{\partial\varphi'}{\partial z} + \mu_0 b H_{0z}\mathbf{H}_0\nabla\varphi'\right\} = 0$$

$$v_z' = \frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + \mathbf{v}_0\nabla\zeta$$

$$\left\{p' + mT' + \frac{1}{4\pi}\mu_0\mathbf{H}_0\nabla\varphi' - a\frac{H_{0z}^2}{4\pi}\mathbf{H}_0\nabla\varphi' - \frac{H_{0z}^2}{4\pi}\mu_T T' - \frac{\mu_0 H_{0z}}{2\pi}\frac{\partial\varphi'}{\partial z}\right\} = -\alpha\Delta\zeta$$

$$a = \frac{1}{H_0}\left(\frac{\partial\mu}{\partial H}\right)_T, \qquad b = \frac{a}{\mu_0} - \frac{\mu_T^2}{4\pi\mu_0 q}, \qquad m = \int_0^{H_0}\mu_T H\, dH$$

$$q = s_T + \frac{1}{4\pi}\int_0^{H_0}\left(\frac{\partial^2\mu}{\partial T^2}\right)H\, dH, \quad \mu_T = \left(\frac{\partial\mu}{\partial T}\right)_H, \quad s_T = \left(\frac{\partial s}{\partial T}\right)_H$$

where $\zeta = \zeta(x, y, t)$ denotes a small displacement of the points of the discontinuity at the z-axis.

The subscript 0 denotes the constant values of the quantities corresponding to the unperturbed motion, and the prime denotes a small perturbation in the corresponding quantity. Henceforth we shall delete the subscript 0 accompanying the equilibrium values. The boundary values in the linear approximation from the pressure, velocity and magnetic field potential will be taken at $z = 0$.

We seek the small perturbations in the form of a plane wave

$$\zeta \sim \exp(i\mathbf{kr} - i\omega t), \quad \mathbf{v}' \sim \exp(i\mathbf{kr} + i\varkappa z - i\omega t) \tag{5}$$

$$p' + mT' \sim \exp(i\mathbf{kr} + i\varkappa z - i\omega t)$$

$$\varphi' \sim T' \sim \exp(i\mathbf{kr} + i\lambda z - i\omega t)$$

where $\mathbf{k}$ is a real vector parallel to the plane $z = 0$ and ω denotes the complex frequency. The parameters λ and $\varkappa$ are also complex, and their imaginary parts represent the coefficients of decay of the corresponding quantities along the z-axis.

Substituting the expressions for T' and φ' from (5) into the third equation of (4), we obtain

$$(\omega - \mathbf{vk})\left[qT' + \frac{i\mu_T}{4\pi}(\mathbf{Hk} + \lambda H_z)\varphi'\right] = 0 \tag{6}$$

If the frequency ω has a positive imaginary part ($\mathrm{Im}\,\omega > 0$), then the perturbation amplitude will increase with time according to the law $\sim \exp(\mathrm{Im}\omega)\,t$, and this will make the flow unstable. The frequency $\omega = \mathbf{vk}$ corresponding to the perturbation moving together with the fluid is real and of no interest from the point of view of the instability, therefore we can write the formula (6) in the form

$$T' = -\frac{i\mu_T}{4\pi q}(\mathbf{Hk} + \lambda H_z)\varphi' \tag{7}$$

Substituting the expressions (5.) and (7) into the fourth equation of (4), we obtain a quadratic equation in λ. Its solution is

$$\lambda = -\frac{bH_z\mathbf{Hk}}{1 + bH_z^2} \pm i\frac{\sqrt{k^2 + b[(\mathbf{Hk})^2 + k^2H_z^2]}}{1 + bH_z^2} \tag{8}$$

Since the only physically meaningful solutions are those which vanish at infinity along the z-axis, we take the minus sign in the region $z < 0$ and the plus sign in the region $z > 0$. Taking div of the first equation of (4) and remembering that $\mathrm{div}\,\mathbf{v}' = 0$, we obtain

$$\Delta(p' + mT') = 0$$

from which it follows that $\varkappa = \pm ik$. Following the choice of signs in the expression for λ, we shall assume that $\varkappa_1 = -ik$ and $\varkappa_2 = +ik$.

From the first and the seventh equation of (4) follows

$$p' + mT' = -\frac{i\rho\zeta}{\varkappa}(\omega - \mathbf{vk})^2 \tag{9}$$

Further, from the boundary conditions for φ' in (4) we obtain the following expressions for $z = 0$:

$$\varphi'_{1(2)} \frac{(\mathbf{Hk})(\mu_2 - \mu_1) + A_{2(1)}(H_{2z} - H_{1z})}{A_2 - A_1} \tag{10}$$

$$A_{1(2)} - \mu_{1(2)}\left[\left(1 + b_{1(2)}\frac{B_z^2}{\mu_{1(2)}^2}\right)\lambda_{1(2)} + b_{1(2)}\frac{B_z}{\mu_{1(2)}}\mathbf{Hk}\right], \quad B_z = \mu H_z$$

Substitution of (7), (9) and (10) into the last equation of (4) produces the dispersion relation

$$\rho_1(\omega - \mathbf{v}_1\mathbf{k})^2 + \rho_2(\omega - \mathbf{v}_2\mathbf{k})^2 - \alpha k^3 + F = 0 \tag{11}$$

$$F = \frac{k(\mu_2 - \mu_1)^2}{4\pi(\mu_1 D_1 + \mu_2 D_2)}\left[\frac{B_z^2}{\mu_1\mu_2}D_1 D_2 - (\mathbf{Hk})^2\right]$$

$$D_{1(2)} = \sqrt{k^2 + b_{1(2)}\left[(\mathbf{Hk})^2 + \frac{B_z^2}{\mu_{1(2)}^2}k^2\right]}$$

The condition of stability of the tangential discontinuity (roots of the quadraric equation (11) are real), has the form

$$\alpha k^3 - \frac{\rho_1\rho_2(\mathbf{uk})^2}{\rho_1 + \rho_2} - F \geqslant 0 \tag{12}$$

where $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$ is the velocity difference between the two fluids. If $\mu = \mathrm{const}$, the condition (12) becomes

$$\frac{(\mu_2 - \mu_1)^2}{4\pi(\mu_1 + \mu_2)}\left[(\mathbf{Hk})^2 - \frac{k^2B_z^2}{\mu_1\mu_2}\right] - \frac{\rho_1\rho_2(\mathbf{uk})^2}{\rho_1 + \rho_2} + \alpha k^3 \geqslant 0 \tag{13}$$

If the fluid is in the gravity field $\mathbf{g}$ (we assume that $\mathbf{g}$ acts in the direction opposing the z-axis), we can show that the term $(\rho_1 - \rho_2) gk$ must be added to the left-hand sides of the expressions (12) and (13).

If the condition (12) is not fulfilled, Eq. (11) has two complex conjugate roots, one of which has a positive imaginary part, and this leads to instability. From the conditions (12) and (13) it follows that the tangential component of the magnetic field stabilizes the discontinuity, since the term $(\mathbf{Hk})^2$ appears in these inequalities with a plus sign. The transverse component H_z destabilizes the discontinuity.

Since a small perturbation on the surface of discontinuity can be represented as a superposition of plane waves, we must find the condition of stability which does not depend on the magnitude and direction of the vector $\mathbf{k}$. In the general case, difficulties of mathematical nature are encountered in establishing this condition, but occasionally it can be obtained.

Let us consider the case $\alpha = 0$, $\mu = \text{const}$, $\mathbf{g} = 0$. The inequality (13) can be written in the form

$$\left[\frac{(\mu_2 - \mu_1)^2}{4\pi(\mu_1 + \mu_2)} H_i H_j - \frac{(\mu_2 - \mu_1)^2 B_z^2}{4\pi\mu_1\mu_2(\mu_1 + \mu_2)} \delta_{ij} - \frac{\rho_1\rho_2}{\rho_1 + \rho_2} u_i u_j\right] k_i k_j \geqslant 0$$

From this quadratic form we obtain the following two conditions of stability:

$$\frac{(\mu_1 - \mu_2)^2}{4\pi(\mu_1 + \mu_2)} \mathbf{h}^2 - \frac{B_z^2 (\mu_1 - \mu_2)^2}{2\pi\mu_1\mu_2(\mu_1 + \mu_2)} - \frac{\rho_1\rho_2 u^2}{\rho_1 + \rho_2} \geqslant 0$$

$$\frac{(\mu_1 - \mu_2)^4}{16\pi^2\mu_1\mu_2(\mu_1 + \mu_2)^2} \left(\frac{B_z^4}{\mu_1\mu_2} - \mathbf{h}^2 B_z^2\right) +$$

$$\frac{(\mu_1 - \mu_2)^2 \rho_1\rho_2}{4\pi(\mu_1 + \mu_2)(\rho_1 + \rho_2)} \left(\frac{B_z^2 u^2}{\mu_1\mu_2} - [\mathbf{uh}]^2\right) \geqslant 0 \quad (\mathbf{h} = (H_{0x}, H_{0y}, 0))$$

In the case when the tangential magnetic field component is absent, the condition of unconditional stability assumes the form

$$[\rho_1\rho_2 u^2 / (\rho_1 + \rho_2) + M]^2 \leqslant 4\alpha g (\rho_1 - \rho_2) \tag{14}$$

$$M = \frac{B_z^2 (\mu_2 - \mu_1)^2 (1 + b_1 H_{1z}^2)^{1/2} (1 + b_2 H_{2z}^2)^{1/2}}{4\pi\mu_1\mu_2 [\mu_1 (1 + b_1 H_{1z}^2)^{1/2} + \mu_2 (1 + b_2 H_{2z}^2)^{1/2}]}$$

When the magnetic field is absent, the inequality (14) becomes the usual hydrodynamic condition of stability given in [1].

If $u = b_1 = b_2 = 0$, the condition (14) yields the following expression for the critical magnitude of the field:

$$B_{kz}^4 = 64\pi^2\mu_1^2\mu_2^2 (\mu_1 + \mu_2)^2 (\mu_2 - \mu_1)^{-4} (\rho_1 - \rho_2) \alpha g$$

The quantity B_{kz} agrees with that obtained in [5].

All the above results can be applied to dielectric fluids by replacing μ by the electric permeability ε and the field $\mathbf{H}$ by the electric field $\mathbf{E}$.

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APPLICATION OF THE BURGERS' EQUATION WITH A VARIABLE COEFFICIENT TO THE STUDY OF NONPLANAR WAVE TRANSIENTS

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We consider the problem of constructing a model equation, i. e. the Burgers' equation, for the wave processes in a thermoelastic medium in the presence of cylindrical and spherical symmetry, and give a solution to the boundary value problem for the initial system of equations.

The Burgers' equation [1 – 4] serves as the model equation for a medium with dissipative properties. A solution of the Burgers' equation describing the motion in the Cartesian coordinate system was studied in detail in [5]. The cases of cylindrical and spherical symmetry however present definite difficulties.

1. Derivation of Burgers' equation with variable coefficients. We consider a process of deformation characterized by the relations

$$x^1 = X^1 + u^1 (X^1,\ t), \quad x^2 = X^2, \quad x^3 = X^3$$

where X^k and x^k $(k = 1, 2, 3)$ are the Lagrangian and Eulerian variables, respectively. The initial equations consist of the laws of conservation of impulse and energy for a continuum, written in a differential form in the Eulerian variables [6, 7]. We write these equations in the Lagrangian coordinates, taking into account the relations connecting the expressions for the physical quantities in the Lagrangian and Eulerian variables, respect-